Section 1.0 - Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If
1. P is true and
2. The truth of P_n implies the truth of P_{n+1} for every positive integer, then P_n is true for all positive integers n.

Example:
Prove that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

General:
$A = n+(n-1)+\ldots+1$
$2A = (1+n)+(1+n)+\ldots+(1+n)$
So $2A = (1+n)+(1+n)+\ldots+(1+n)$
Therefore, $2A = n(1+n)$
Consequently, $A = \frac{n(n+1)}{2}$

Proof by induction:
$P_n = 1+2+\ldots+n = \frac{n(n+1)}{2}$
1. $n = 1$ is P_1 true?
$1? = \frac{1(1+1)}{2} = 1$ so P_1 is true
2. Suppose P_k is true. Prove for P_{k+1}
$P_{k+1} = 1+2+\ldots+(k+1) = \frac{(k+1)(k+2)}{2}$
The assumption was that $1+2+\ldots k = \frac{k(k+1)}{2}$

Proof by contradiction:
Prove that $\sqrt{2}$ is irrational
Suppose $\sqrt{2} = \frac{p}{q}$ where $p,q \in \mathbb{Z}$ and $q \neq 0$ and $\frac{p}{q}$ is not reducable
$2 = \frac{p^2}{q^2}$, $p^2 = 2q^2$
p^2 must be even
p must also be even
There exists an integer k such that $p = 2k$
Therefore, $4k^2 = 2q^2$ and $2k^2 = q^2$
Then, p and q must both be even
But if that is the case, $\frac{p}{q}$ is reducable.

1.1 - Introduction to Systems of Linear Equations

A system of m linear equations in n variables is a set of m equations, each of which is linear in the same n variables.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3$
...
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$

a_{ij} = (ith equation, jth variable)

Example:
$x + y + z = 3$
$2x + 4y + 5z = 7$
$5x - 7y - 5z = 2$

Definition: A system of linear equations is *consistant* if it has at least one solution, and it is *inconsistant* if it has no solution.

There are three cases:
1. The system has exactly one solution (consistant system)
2. The system has an infinite number of solutions (consistant system)
3. The system has no solutions (inconsistant system)

Example:
a) $x + y = 3$ and $x - y = -1$
$(1) + (2) \Rightarrow 2x = 2 \Rightarrow x = 1$
from (1) we have $y = 3 - x = 3 - 1 = 2$
Solution: $(1, 2)$

b) $x + y = 3$ and $2x + 2y = 6$
There are infinately many solutions because they are the same equation

c) $x + y = 3$ and $x + y = 1$
There are no solutions. These equations are parallel lines.

$2x + 7y = 5$
$3y = 4$
This is in row-echelon form. To solve it, use *back substitution*.

$x - 2y + 3z = 9$

$-x + 3y = -4$
$2x - 5y + 5z = 17$

Elementary row operations:
1. Two equations may be interchanged.
2. Both sides of an equation may be multiplied by a nonzero constant.
3. A multiple of an equation may be added to another equation.

Step 1:
$x - 2y + 3z = 9$
$y + 3z = 5$
$2x - 5y + 5z = 17$

Step 2:
$x - 2y + 3z = 9$
$y + 3z = 5$
$-y - z = -1$

Step 3:
$x - 2y + 3z = 9 \Rightarrow x = 9 + 2y - 3z$
$y + 3z = 5 \Rightarrow y = 5 - 3z = -1$
$2z = 4 \Rightarrow z = 2$

The method above used to put the equation in row-echelon form is called *Gaussian Elimination*.

1.2 - Gaussian Elimination and Gauss-Jordan Elimination

$x + y = 3$
$2x + 2y = 6$
$-2(1) + (2) \Rightarrow x + y = 3$ and $0 = 0$
Consistant system

Definition: If m and n are positive integers, then an $m \times n$ *matrix* is a rectangular array in which each *entry*, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m *rows* (horizontal lines) and n *columns* (vertical lines).

A matrix is said to be a *square matrix of order n* if $m = n$. The entries a_{11}, a_{22}, a_{nm} are called the main diagonal entries.

Size 3×3: $\begin{matrix} 1 & -1 & \pi \\ 0 & 2 & 5 \\ 3 & 1 & e \end{matrix}$

The main diagonals of this matrix are: $1, 2, e$

Size 1×4: $\begin{matrix} 1 & 3 & \sqrt{2} & -1 \end{matrix}$

Size 3×1: $\begin{matrix} 1 \\ 2 \\ \pi \end{matrix}$

A system of linear equations can be represented by using a matrix.

Example:

$5x + 3y - z = 2$

$2x + y - 2z = 0$

$2x - z = 1$

Coefficient matrix:

$$\begin{matrix} 5 & 3 & -1 \\ 2 & 1 & -2 \\ 2 & 0 & -1 \end{matrix}$$

Augmented Matrix

$$\begin{matrix} 5 & 3 & -1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 0 & -1 & 1 \end{matrix}$$

Elementary Row Operations (Gaussian Elimination with back substitution):

Two matricies are said to be row-equivalent if one can be obtained from the other by a sequence of elementary row operations.

1. Two equations may be interchanged.
2. Both sides of an equation may be multiplied by a nonzero constant.
3. A multiple of an equation may be added to another equation.

Associated Augmented Matrix:

$$\begin{matrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{matrix} \Rightarrow \begin{matrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{matrix} \Rightarrow \begin{matrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{matrix} \Rightarrow \begin{matrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 3 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{matrix} \Rightarrow \begin{matrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 15 \end{matrix} \text{, Gaussian elimination: } \begin{matrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & -5 \end{matrix}$$

The last line tells us that $0 = 15$. This is a contradiction, and therefore this is an inconsistant system with no solutions.

Gauss-Jordan Elimination:

A matrix in row-echelon form is in reduced row echelon form if every column that has a leading 1 has 0s in every position above and below its leading 1.

$$\begin{matrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Let us use Gauss-Jordan elimination to solve the system:

$x - 2y + 3z = 9$

$-x + 3y = -4$

$2x - 5y + 5z = 17$

Gaussian Elimination:

$$\begin{matrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{matrix}$$

Jordan Elimination:

$$\begin{matrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{matrix}$$

1.3 - Applications of Systems of Linear Equations

Suppose a collection of data is represented by n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy plane, and you are asked to find a polynomial function $p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ of degree $n - 1$ whose graph passes through the given points. This procedure is called *polynomial curve fitting*.

Example: Determine the polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ whose graph passes through the points $(1,4), (2,0), (3,12)$

$a_0 + 1a_1 + a_2 = 4$
$a_0 + 2a_1 + 4a_2 = 0$
$a_0 + 3a_1 + 9a_2 = 12$

$a_0 = 24$
$a_1 = -28$
$a_2 = 8$