2.1 - Operations with Matricies

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matricies of size $m \times n_1$ then their sum is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$.

Example:

$$\begin{matrix} 1 & 2 \\ 3 & 9 \end{matrix} + \begin{matrix} -1 & 0 \\ 0 & 3 \end{matrix} = \begin{matrix} 0 & 2 \\ 3 & 12 \end{matrix}$$

If c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix given by $cA = [ca_{ij}]$.

Example:

$$2 * \begin{matrix} 1 & 3 \\ -1 & 4 \end{matrix} = \begin{matrix} 2 & 6 \\ -2 & 8 \end{matrix}$$

If $A = [a_{ij}]$ is an $m \times n$ matrix, and $B = [b_{ij}]$ is a $n \times p$ matrix, then the product $A * B$ will be a $m \times p$ matrix $A * B = [c_{ij}]$

$$c_{ij} = \Sigma_{k=1}^{n} a_{i3} b_{kj} = a_{i2} + b_{ij} + a_{i3} b_{ij} + \ldots + a_{ij} b_{nj}$$

Example 1:

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} * \begin{matrix} -1 & 2 \\ 3 & 1 \end{matrix} = \begin{matrix} (1 * -1) + (2 * 3) & (1 * 2) + (2 * 1) \\ (3 * -1) + (4 * 3) & (3 * 2) + (4 * 1) \end{matrix} = \begin{matrix} 5 & 4 \\ 9 & 10 \end{matrix}$$

Example 2:

$$\begin{matrix} 1 & 2 & 3 \end{matrix} * \begin{matrix} 2 & 4 \\ 1 & -1 \\ 0 & 2 \end{matrix} = \begin{matrix} 2 + 2 + 0 & 4 - 2 + 6 \end{matrix} = \begin{matrix} 4 & 8 \end{matrix}$$

2.2 - Properties of Matrix Operations

If A, B, C are $m \times n$ matricies and c and d are scalars, then the following properties are true:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $(c * d) * A = c * (d * A)$
4. $1 * A = A$

5. $c * (A + B) = cA + cB$
6. $(c + d)A = cA + dA$

If the size of those matricies are such that the given matrix products are defined, then:
1. $A(BC) * (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

Example: Non-commutativity of matrix multiplication
$AB \neq BA$ all the time.

$$A = \begin{matrix} 1 & 3 \\ 2 & -1 \end{matrix}$$

$$B = \begin{matrix} 2 & -1 \\ 0 & 2 \end{matrix}$$

$$AB = \begin{matrix} 2 & 5 \\ 4 & -4 \end{matrix}$$

$$BA = \begin{matrix} 0 & 7 \\ 4 & -2 \end{matrix}$$

Identity matrix of order 3:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

For repeated multiplication of square matricies, $A^k = A * A\ldots A$ (k factors)
define $A^0 = I_n$, $A^jA^k = A^{j+k}$; $(A^j)^k = A^{jk}$

The transpose of a matrix is formed by switching the rows and columns of a matrix
$A = [a_{ij}]$ then $A^T = [b_{ij}]$ where $b_{ij} = a_{ij}$

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}^T = \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

$A = [a_{ij}]$ then $A^T = [b_{ij}]$ where $b_{ij} = a_{ij}$
1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = A^T B^T$

4. Proof
Let $A = [a_{ij}]$ and $B = [b_{ij}]$
$AB = C$
$c_{ij} = \sum_{k=1}^{n} a_{ij} b_{kj}$
$(AB)^T = D$
$d_{ij} = \sum_{k=1}^{n} a_{jk} b_{ki}$
$B^T A^T = E$
$e_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk}$
Therefore $D = E$
So $(AB)^T = B^T A^T$

If $A = A^T$, then A is called a symmetric matrix.
Example: I_n is symmetric
Let us prove that $B = A * A^T$ is symmetric.
Proof:
$B^T = (A * A^T)^T = (A^T)^T A^T = A * A^T = B$
Therefore, AA^T is symmetric.

2.3 - The Inverse of a Matrix

Definition:
An $n \times n$ matrix A is invertible (or nonsingular) if there exists an $m \times n$ matrix B such that $AB = BA = I_n$.

B is called the inverse of A. A matrix that does not have an inverse is called noninvertible (or singular).

Note: Nonsquare matricies do not have inverses.

Uniqueness of an inverse matrix:
If A is an invertable (nonsingular) matrix, then the inverse is unique.

The inverse of matrix A is denoted by A^{-1} or Inv(A).

Proof:
Suppose A has 2 inverses, B & C.
$AB = BA = I_n$

$AC = CA = I_n$
$A = I_n$
Multiply by C from the left $\Rightarrow$
$CAB = CI_n = C$
$I_nB = C$
So $B = C$

$AA^{-1} = A^{-1}A = I_n$

Example:

$$A = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$

$$B = \begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}$$

$AB = BA = I_2$

Finding the inverse of a matrix by Gauss-Jordan elimination
Let A be a square matrix of order n.
1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I_n on the right to obtain $[A|I_n]$
2. If possible, row reduce A to I_n using elementary row operations on the entire matrix $[A|I_n]$. The result will be the matrix $[I_n|A^{-1}]$. If this is not possible, then the matrix has no inverse and A is not invertable.

Example: $A = \begin{matrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{matrix}$

1. $\begin{matrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{matrix}$

$(2) = -1(1) + (2)$
$(3) = 6(1) + (3)$

$$\begin{matrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{matrix}$$

$(3) = 4(2) + (3)$

$(3) = -(3)$

$$\begin{matrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{matrix}$$

$(2) = (3) + (2)$

$$\begin{matrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{matrix}$$

$(1) = (2) + (1)$

$$\begin{matrix} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{matrix}$$

So A^{-1} is

$$\begin{matrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{matrix}$$

$(2) = -3(1) + (2)$

$$\begin{matrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{matrix}$$

$(1) = (2) + (1)$

$$\begin{matrix} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{matrix}$$

$(2) = -\frac{1}{2}(2)$

$$\begin{matrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{matrix}$$

$$A^{-1} = \begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}$$

Properties of the inverse:

If A is an invertable matrix, and k is a positive integer, and c is a scalar, then:

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1} * A^{-1} \ldots A^{-1}$ (k times)
3. $(cA)^{-1} = \frac{1}{c} A^{-1}$ for $c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

Theorem: If A and B are invertable matrices of size n, then AB is invertable and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $AB * B^{-1}A^{-1}$
$B * B^{-1} = I$
$AB * B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$
$B^{-1}A^{-1} * AB = B^{-1}IB = B^{-1}B = I$

Cancellation Properties:
If C is an invertable matrix, then
1. If $AC = BC$ then $A = B$
Proof: $AC = BC$ implies $ACC^{-1} = BCC^{-1}$
2. If $CA = CB$ then $A = B$

Systems of equations with unique solutions:
If A is an invertable matrix, then the system of linear equations $Ax = b$ has a unique solution given by $x = A^{-1}b$

Example:
$x + 2y = 5$
$3x + 4y = -2$

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} * \begin{matrix} x \\ y \end{matrix} = \begin{matrix} 5 \\ -2 \end{matrix}$$

$$\begin{matrix} x \\ y \end{matrix} = A^{-1} \begin{matrix} 5 \\ -2 \end{matrix}$$

$Ax = b$
If A is invertable, then $x = Ab$

2.4 - Elementary Matrices

Definition: An $n \times n$ matrix is called an *elementary matrix* if it can be obtained from I_n by a single elementary row operation

Example: $\begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix}$ is an elementary matrix

$$\begin{matrix} 1 & 0 \\ 2 & 1 \end{matrix} \quad \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \quad \begin{matrix} -1 & 0 \\ 0 & 2 \end{matrix}$$

Look at the following multiplication

$$\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix} \times \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} = \begin{matrix} 1 & 2 \\ 6 & 8 \end{matrix}$$

LU (Lower/Upper) Factorization

Example:

Lower: $\begin{matrix} 1 & 0 \\ 3 & 1 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 6 \end{matrix}$

Upper: $\begin{matrix} 1 & 2 \\ 0 & 3 \end{matrix} \quad \begin{matrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{matrix}$

If we can write $A = LU$ where L is lower triangular and U is upper triangular, then $LUx = b$

Let $Ux = y$
So $Ly = b$
We solve for y easily using forward substitution.
So we go back to $Ux = y$ and solve easily using back substitution.

$x + 2y = 3$
$x = 4$

$$\begin{matrix} 1 & 2 \\ 1 & 0 \end{matrix} \times \begin{matrix} x \\ y \end{matrix} = \begin{matrix} 3 \\ 4 \end{matrix}$$

$$A = \begin{matrix} 1 & 2 \\ 1 & 0 \end{matrix} \; (2) = -1(1) + 2 \rightarrow \begin{matrix} 1 & 2 \\ 0 & -2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

$E_1 \times A = U \Rightarrow A = E_1^{-1} U$ and $E_1^{-1} = L$

$$L = E_1^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} =$$

Shortcut:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ where $(ad - bc \neq 0)$

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$