5.1 - Length and Dot Product in R^n

Def: The length of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is given by
$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2 + \ldots + (v_n)^2}$
If $||\vec{v}|| = 1$, then the vector $\vec{v}$ is called a unit vector.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and c is a scalar. Then $||c\vec{v}|| = |c|\,||\vec{v}||$

If v is a nonzero vector in $\mathbb{R}^n$, the vector $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$ is a unit vector.
We call $\vec{u}$ the unit vector in the direction of $\vec{v}$
*Proof: $||\vec{u}|| = \left|\left|\frac{\vec{v}}{||\vec{v}||}\right|\right| = \frac{1}{||\vec{v}||} = 1$

Example: Find the unit vector in the direction of the vector $\vec{v} = (1, 1, 1)$
$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

The dot product of $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ is the quantity
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$

Theorem: If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, then
1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u}\left(\vec{v} \cdot \vec{w}\right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c\left(\vec{u} \cdot \vec{v}\right) = \left(c\vec{u}\right) \cdot \vec{v} = \vec{u} \cdot \left(c\vec{v}\right)$
4. $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$
5. $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0 \Leftrightarrow \vec{v} = \vec{0}$

Example: Given that $\vec{u} \cdot \vec{u} = 4$, $\vec{u} \cdot \vec{v} = -1$, $\vec{v} \cdot \vec{v} = 2$. Find $\left(\vec{u} + \vec{v}\right)\left(3\vec{u} - \vec{v}\right)$
$\left(\vec{u} + \vec{v}\right)\left(3\vec{u} - \vec{v}\right) = 3\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + 3\vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 12 + 1 - 3 - 2 = 8$

We define the angle between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ by $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}$ for
$0 \leq \theta \leq \pi$

Example: $\vec{u} = \langle 1, 2 \rangle$; $\vec{v} = \langle 2, 3 \rangle$
$\arccos \frac{(1,2) \cdot (2,3)}{\sqrt{5}\sqrt{13}} = \frac{8}{\sqrt{65}}$
$-1 \leq \cos\theta \leq 1$
$-||\vec{u}|| \cdot ||\vec{v}|| \leq \vec{u} \cdot \vec{v} \leq ||\vec{u}|| \cdot ||\vec{v}||$

Cauchy-Schwarz Inequality: $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}|| \cdot ||\vec{v}||$

Case 1: Suppose $\vec{u} = \vec{0}$. Then $||\vec{u}|| \cdot ||\vec{v}|| = 0$ and $|\vec{u} \cdot \vec{v}| = 0$.

Case 2: If $\vec{u} \neq 0$, let $t \in \mathbb{R}$: $(t\vec{u} + \vec{v}) \cdot (t\vec{u} + \vec{v}) \geq 0$

$t^2(\vec{u} \cdot \vec{u}) + t\vec{u} \cdot \vec{v} + t\vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \geq 0$

$t^2(\vec{u} \cdot \vec{u}) + 2t\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = 0$

However, if we have $at^2 + bt + c \geq 0$ for all $t \in \mathbb{R}$, then $b^2 - 4ac \leq 0$

So $(2\vec{u} \cdot \vec{v})^2 = 4(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) \leq 0$.

Therefore, $4(\vec{u} \cdot \vec{v})^2 \leq 4(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$ and $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}|| \cdot ||\vec{v}||$

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal to each other if their dot product is zero.

Example: $\vec{u} = (1, 1, 1, -1)$; $\vec{v} = (-2, 0, 1, -1)$

$\vec{u} \cdot \vec{v} = 0$

The triangle inequality: If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

$||\vec{u} + \vec{v}||^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$

$= ||\vec{u}||^2 + 2\vec{u} \cdot \vec{v} + ||\vec{v}||^2$

$\leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2$

$= (||\vec{u}|| + ||\vec{v}||)^2$

So $||\vec{u} + \vec{v}||^2 \leq (||\vec{u}|| + ||\vec{v}||)^2$

$||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + 2\vec{u} \cdot \vec{v} + \|v\|^2$

However, if $\vec{u} \cdot \vec{v} = 0$, then $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$ (Pythagorian theorem)

5.2 - Inner Product Spaces

Definition: Let u, v, w be vectors in a vector space V, and let c be any scalar.

An inner product on V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors u and v, and satisfies the following:

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $c\langle u, v \rangle = \langle cu, v \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$

A vector space V with an inner product is called an inner product space.

Example: Take $V = \mathbb{R}^2$ and $\langle u, v \rangle = u_1v_1 + 2u_2v_2$ where $u = \langle u_1, v_2 \rangle$; $v = \langle v_1, v_2 \rangle$

Take $V = M_{2,2}$ and $\langle A,B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Take $V = C[a,b]$ where $\langle f,g\rangle =$

$f(a) * g(a)$? No, $f(0) * g(0) = 0$, but a is not necessarily 0.

$\langle f,g\rangle = \int_a^b f(x) * g(x)dx$

$\langle f,f\rangle = 0 \Rightarrow \int_a^b [f(x)]^2 dx = 0$

Definition: Let u and v be vectors in an inner product space V.

1. The norm (or length) of u is $||u|| = \sqrt{\langle u,u\rangle}$
2. The angle between two nonzero vectors u,v is given by $\cos\theta = \frac{\langle u,v\rangle}{||u||*||v||}$ where $0 \leq \theta \leq \pi$.
3. u,v are orthogonal if $\langle u,v\rangle = 0$

Examples:

Let $V = M_{2,2};\ A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix};\ B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

What is $||A||, ||B||$, the angle between A and B?

$\langle A,B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$\arccos \frac{\langle A,B\rangle}{||A||||B||} = \frac{(1*2)+(1*1)+(2*(-1))+(3*0)}{\sqrt{1+1+4+9}\sqrt{4+1+1+0}} = \frac{1}{90}\sqrt{6}\sqrt{15}$

$\arccos \frac{\sqrt{90}}{90} = \arccos \frac{1}{30}\sqrt{10}$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\langle 1,1\rangle,\langle 1,1\rangle} = \sqrt{3}$

$\overrightarrow{proj_{\vec{v}}\vec{u}} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$

For inner product space V:

$proj_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle}v$

Example: Let $f(x) = x$ and $g(x) = x^2$ be functions on $C[0,1]$.

$proj_g f = \frac{\langle f,g\rangle}{\langle g,g\rangle}g = \frac{\int_0^1 f(x)g(x)dx}{\int_0^1 g(x)g(x)dx} = \frac{\int_0^1 x^3dx}{\int_0^1 x^4dx}g = \frac{\frac{1}{4}}{\frac{1}{5}}x^2 = \frac{5}{4}x^2$

Section 5.3 - Orthogonal Bases: Gram-Schmidt Process

Definition: A set S of vectors in an inner product space V is called orthogonal if every pair of vectors in S is orthogonal.

If, in addition, each vector in the set S is a unit vector, then S is called orthonormal.

Suppose $S = \{v_1, v_2, \ldots, v_n\}$

1. $\langle v_i, v_j \rangle = 0$ if $i \neq j$ (orthogonal)
2. $\langle v_i, v_j \rangle = 0$ and $\|v_i\| = 1$ for all i, then it is orthonormal

If S is a basis, then it is called an orthogonal basis or an orthonormal basis, respectively.

In $P_2 : \langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2$, where $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_{1x} + b_2x^2$.

In $P_n : \langle p, q \rangle = \sum_{i=0}^{n} a_ib_i$

For P_n (polynomials degree $\leq n$), a standard basis is $\{1, x, x^2, \ldots, x^{n-1}, x^n\}$.

Show that this is an orthonormal basis:

1. Let $p_i(x) = x^i$ and $p_j(x) = x^j$ $(i \neq j)$

$P_i(x) = 0 + 0x + 0x^2 + \ldots + 0x^{i-1} + 1x^i + 0x^{i+1} + \ldots + 0x^n$

$P_j(x) = 0 + 0x + 0x^2 + \ldots + 0x^{j-1} + 1x^j + 0x^{j+1} + \ldots + 0x^n$

$\langle p_i, p_j \rangle = 0 * 0 + 0 * 0 + 0 * 0 + \ldots + 0 * 0 = 0$

2. Let $P_i(x) = x^i$ then $\|p_i\| = \sqrt{\langle p_i, p_i \rangle} = \sqrt{1} = 1$ for every $i \in \{0, \ldots, n\}$.

So B is an orthonormal basis.

In $C[0, 2\pi] : \langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$

$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(x), \frac{1}{\sqrt{\pi}} \cos(x), \ldots, \frac{1}{\sqrt{\pi}} \sin(nx) \frac{1}{\sqrt{\pi}} \cos(nx) \right\}$

is an orthogonal set.

Theorem: If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

*Proof: $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$

$c_1\langle v_1, v_i \rangle + c_2\langle v_2, v_i \rangle + \ldots + c_i\langle v_i, v_i \rangle + \ldots + c_n\langle v_n, v_i \rangle = \langle 0, v_i \rangle$

$c_i\|v_i\|^2 = 0 \Rightarrow c_i = 0$ for every $i \in \{1, 2, \ldots, n\}$, so it is linearly independent.

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis of V.

Theorem: If $B = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector w with respect to B is
$w = \sum_{i=1}^{n} \langle w, v_i \rangle v_i$

Proof: There exist uniquely $c_1, c_2, \ldots, c_n$ such that $w = \sum_{i=1}^{n} c_i v_i$.
$\langle w, v_i \rangle = \left\langle \sum_{i=1}^{n} c_i v_i, v_i \right\rangle = \sum_{j=1}^{n} \langle c_j, v_j, v_i \rangle = \sum_{j=1}^{n} c_j \langle v_j, v_i \rangle$
$\langle w, v_i \rangle = c_i \|v_i\|^2 = c_i <\!\!- \|v_i\|^2 = 1$ because it is orthonormal.

The c_i coefficients are called the Fourier coefficients.

Theorem: Gram-Schmidt Orthonormalization Process
1. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for an inner product space V.
2. Let $B' = \{w_1, w_2, \ldots, w_n\}$ where w_i is given by:
$w_1 = v_1,\ w_2 = v_2 \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1, \ldots,\ w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \ldots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$
3. Let $u_i = \frac{w_i}{\|w_i\|}$, then $B'' = \langle u_1, u_2, \ldots, u_n \rangle$ is an orthonormal basis for V.

Example 1: $B = \left\{(1,1),\ (0,1)\right\}$
$w_1 = (1,1)$
$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = (0,1) - \frac{\langle (1,1),(0,1) \rangle}{\langle (1,1),(1,1) \rangle}(1,1) = (0,1) - \frac{1}{2}(1,1) = (-\frac{1}{2}, \frac{1}{2})$
$u_1 = \frac{w_1}{\|w_1\|} = \frac{(1,1)}{\sqrt{2}} = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$
$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}} = \left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$
$\{u_1, u_2\}$ is an orthonormal basis.

Example 2: $B = \{1, x, x^2\}$ where $V = P_2$
$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) dx = \frac{2}{3} pq$