Section 3.1 - The Determinant of a Matrix

Definition of the determinant of a 2×2 matrix:
The determinant of the matrix A

$$A = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$$

is given by $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

Definition of Minors and Cofactors of a matrix:
If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A.
The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

Definition of the determinant of a matrix:
If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in any row or any column, multiplied by their cofactors.
That is: $\det(A) = |A| = \sum_{j=1}^{n} a_{cj} = a_{i1}c_{i1} + a_{i2}c_{i2} + a_{i3}c_{i3} + \ldots + a_{in}c_{in}$ <– i^{th} row expansion
or $\det(A) = |A| = \sum_{i=1}^{n} a_{ij}c_{ij} = a_{ij}c_{ij} + a_{2j}c_{2j} + \ldots + a_{nj}c_{nj}$ <– j^{th} row expansion

Example: $A = \begin{matrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{matrix} = [a_{ij}]$ where $1 \leq i \leq 3$ and $1 \leq j \leq 3$

minor of $a_{21} = 3$ is: $M_{21} = \begin{matrix} 2 & 1 \\ 0 & 1 \end{matrix} = 2$

cofactor of $a_{21} = 3$ is: $C_{21} = (-1)^{2+1}\, M_{21} = (-1)^{3} * 2 = -2$

Expansion with respect to the second row:
$\det(A) = |A| = 3C_{21} - 1C_{22} + 2C_{23}$

$$= 3(-2) + (-1)(-1)^{2+2} \begin{matrix} 0 & 1 \\ 4 & 1 \end{matrix} + 2(-1)^{2+3} \begin{matrix} 0 & 2 \\ 4 & 0 \end{matrix} = 3(-2) + 4 + 16 = 14$$

Example:
$+ - + -$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} 6 & 7 & 8 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 5 & 7 & 8 \\ 4 & 2 & 1 \\ 0 & 2 & -1 \end{vmatrix} + 3\begin{vmatrix} 5 & 6 & 8 \\ 4 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 4\begin{vmatrix} 5 & 6 & 7 \\ 4 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix}$$

3×3 shortcut

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} \Rightarrow \begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & 0 \end{matrix} \Rightarrow (0+16+0)-(-4+0+6) = 14$$

Theorem: Determinant of a triangular matrix:

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal.

That is, the $\det(A) = a_{11} + a_{22} + a_{33} + a_{mm}$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \det(A) = 8$

Section 3.2 - Evaluation of a Determinant Using Elementary Operations

Let A and B be square matrices.

1. If B is obtained from A by interchanging two rows or two columns of A, then $\det(B) = -\det(A)$
2. If B is obtained from A by adding a multiple of a row or column of A to another row or column of A, then $\det(B) = \det(A)$
3. If B is obtained from A by multiplying a row or a column of A by a non-zero constant c, then $\det(B) = c * \det(A)$
4. If A is an $n \times n$ matrix, and c is a nonzero scalar, then $\det(cA) = c^n \det(A)$

Theorem: If the determinant of a matrix is zero, then it is invertable

The determinant of the matrix is 0 if:

1. An entire row or column consists of zeros
2. Two rows or columns are equal
3. One row or column is a multiple of another row or column

Section 3.3 - Properties of Determinants

If A and B are square matrices of order n and c is a scalar, then

1. $\det(AB) = \det(A)\det(B)$
2. $\det(cA) = c^n \det(A)$

Example:

$$A = \begin{matrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{matrix}$$

$$A' = \begin{matrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 10 \end{matrix}$$

$A = 10A'$
$\det(A') = 5$
$\det(A) = 1000$

Note that $\det(A + B) \neq \det(A) + \det(B)$

Important theorem: A square matrix A is invertable if and only if $\det(A) \neq 0$
Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$
Proof: A is invertable. There is an A^{-1} such that
$AA^{-1} = A^{-1}A = I \Rightarrow \det(AA^{-1}) = I \Rightarrow \det(A)\det(A^{-1}) = I$
So $\det(A^{-1}) = \frac{1}{\det(A)}$

If A is a square matrix, then $\det(A) = \det(A^T)$

Section 3.5 - Applications of Determinants

1. Cramer's Rule:
$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

$$x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

Example:

$(\cos\theta)x + (\sin\theta)y = 1$

$(-\sin\theta)x + (\cos\theta)y = 1$

$$x = \frac{\det\begin{pmatrix} 1 & \sin\theta \\ 1 & \cos\theta \end{pmatrix}}{\det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} = \frac{\cos\theta - \sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$y = \frac{\det\begin{pmatrix} \cos\theta & 1 \\ -\sin\theta & 1 \end{pmatrix}}{\det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} = \frac{\cos\theta + \sin\theta}{\cos^2\theta + \sin^2\theta}$$

Solution: $(\cos\theta - \sin\theta, \cos\theta + \sin\theta)$

General case:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots$

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$

$$x_i = \frac{\det\begin{pmatrix} a_{11} & a_{1(i-1)} & b_1 & b_{1(i-1)} & a_n \\ a_{21} & a_{2(i-2)} & b_2 & b_{2(i-2)} & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n(i-n)} & b_n & b_{n(i-n)} & a_{nn} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}}$$

2. Area of a triangle in the xy plane

The area of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by:

$$A = \pm\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Corollary: Three points are colinear if the area of the triangle is zero.

The equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

The volume of the tetrahedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is given by

$$\text{Volume} = \pm\frac{1}{6}\begin{matrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{matrix}$$

Corollary: Four points are coplanar if and only if the volume of them is zero.