Section 4.1 - Vectors in R^n

A vector in the plane is represented geometrically by a directed line segiment whose initial point is the origin, and whose terminal point is the point (x, y).

Properties of Vectors: Let $\vec{u}_1\vec{v}_1$ and $\vec{w}$ be vectors in the plane, and let c and d be scalars.

1. $\vec{u} + \vec{v}$ is a vector in the plane.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right)$
4. $\vec{u} + \vec{0} = \vec{u}$
5. $\vec{u} + \left(-\vec{u}\right) = \vec{0}$
6. $c\vec{u}$ is a vector in the plane
7. $c\left(\vec{u} + \vec{v}\right) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c\left(d\vec{u}\right) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2 \ldots, v_n)$ be vectors in $\mathbb{R}^n$ and let c be a real number.

Then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$
and $c\vec{u} = (cu_1, cu_2, \ldots, cu_n)$

The same 10 properties apply to $\mathbb{R}^n$.

Example:
$\vec{u} = (0, 5, -2, 1)$ and $\vec{v} = (3, 4, 1, -1)$ and $c = -2$
$c\vec{u} + v = (3, -6, 5, -3)$

To write a vector $\vec{x}$ as a linear combination of the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_n}$, we need to find scalars c_1, c_2, and c_n such that
$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} = \sum_{i=1}^{n} c_i v_i$

Example: Let $\vec{x} = (-1, -2, -2)$ and $\vec{u} = (0, 1, 4)$ and $\vec{v} = (-1, 1, 2)$ and $\vec{w} = (3, 1, 2)$. Find scalars a, b, and c such that $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$
$(-1, -2, -2) = (0, a, 4a) + (-b, b, 2b) + (3c, c, 2c)$
So $(-1, -2, -2) = (-b + 3c, a + b + c, 4a + 2b + 2c)$

$$\begin{matrix} 0 & -1 & 3 & -1 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 2 & -2 \end{matrix} \text{, row echelon form: } \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{matrix}$$

Section 4.2 - Vector Spaces

Definition of a vector space: Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every element u, v and w and every scalar (real number) c and d, then V is called a vector space and the elements are called vectors.

Addition:
1. $\vec{u} + \vec{v}$ is in V
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + \left(\vec{v} + \vec{w}\right) = \left(\vec{u} + \vec{v}\right) + \vec{w}$
4. V has a zero vector 0 such that for every $\vec{u}$ in V, $\vec{u} + 0 = \vec{u}$
5. For every $\vec{u}$ in V, there is a vector in V denoted by $-\vec{u}$ such that $\vec{u} + \left(-\vec{u}\right) = 0$

Scalar Multiplication
6. $c\vec{u}$ is in V.
7. $c\left(\vec{u} + \vec{v}\right) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c\left(d\vec{u}\right) = (cd)\vec{u}$
10. $1\left(\vec{u}\right) = \vec{u}$

Some important vector spaces:
$\mathbb{R}$ = the set of all real numbers
$\mathbb{R}^2$ = the set of all ordered pairs
$\mathbb{R}^3$ = the set of all ordered triples
$\mathbb{R}^n$ = the set or all ordered n-tuples
$C(-\infty, \infty)$ = the set of all continous functions defined on the real line.
$C[a, b]$ = the set of all continous functions defined on the closed interval $[a, b]$
P = the set of all polynomials
P_n = the set of all polynomials of degree $\leq n$
$M_{m,n}$ = the set of all $m \times n$ matrices
$M_{n,n}$ = the set of all $n \times n$ square matrices

Sets that are not vector spaces
The set of integers

The set of nth degree polynomials

Example 1:
$p(x) = x^3 + x^2$
$q(x) = -x^3 + x$
$p(x) + q(x) = x^2 + x$ <– Failure of property 1

Example 2:
Let $V = \mathbb{R}^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the following nonstandard definition of scalar multiplication:
$c(x_1, x_2) = (cx_1, 0)$
10. $1\vec{u} = \vec{u}$
$1(x_1, y_1) = (1x_1, 0)$

Example 3:
The set of all $n \times n$ singular matrices with the standard operations is not a vector space.

$$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

There are cases where two singular matrices, s and t, when added will produce a nonsingular matrix n.

Section 4.3 - Subspaces of Vector Spaces

Definition: A nonempty subset W of a vector space V is called a subspace of V if W is itself a vector space under the operations of addition and scalar multiplication defined in V. $(W \in V)$

Test for a subspace: If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold:
1. If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u} + \vec{v} \in W$
2. If $\vec{u} \in W$ and c is a scalar, then $c\vec{u} \in W$

Example: Let W be the set of all 2×2 symmetric matrices.
$W \subset M_{2,2}$, which is a vector space
1. Let $A, B \in W$. $(A+B)^T = A^T + B^T = A + B$. Therefore, $A + B$ is symmetric, and $A + B \in W$.
2. Let $A \in W$ and $c \in \mathbb{R}$. $(cA)^T = cA^T = cA$. Therefore, $cA \in W$

Theorem: If V and W are both subspaces of a vector space U, then the intersection of V and W, denoted by $V \cap W$, is also a subspace of U.

$V \cap W \subset U$

1. Let $\vec{u}, \vec{v} \in V \cap W$. Then $\vec{u}, \vec{v} \in V$ and $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} = V$ and $\vec{u} + \vec{v} = W$. Therefore, $\vec{u} + \vec{v} \in V \cap W$.

2. Let $\vec{u} \in V \cap W$ and $c \in \mathbb{R}$. Then $\vec{u} \in V$ and $\vec{u} \in W$. $c\vec{u} \in V$ and $c\vec{u} \in W \Rightarrow c\vec{u} \in V \cap W$.

What about the union of two subspaces?

$V = \{(x, 0)$ where $x \in \mathbb{R}\}$

$W = \{(0, y)$ where $y \in \mathbb{R}\}$

$(1, 0) \in V \cup W$

$(0, 1) \in V \cup W$

But $(1, 0) + (0, 1) = (1, 1)$ and $V \cup W$. So it is not a subspace of $\mathbb{R}^2$.

Section 4.4 - Spanning Sets and Linear Independence

A vector $\vec{v}$ in a vector space V is called a linear combination of the vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}$ if $\vec{v}$ can be written in the form $\vec{v} = c_1\vec{u_1} + c_2\vec{u}_2 + \ldots + c_k\vec{u_l}$ where $c_1, c_2, \ldots, c_k$ are scalars.

Example: $V = M_{2,2}$

$$\vec{v} = \begin{matrix} 0 & 8 \\ 2 & 1 \end{matrix}, \vec{u_1} = \begin{matrix} 0 & 2 \\ 1 & 0 \end{matrix}, \vec{u_2} = \begin{matrix} -1 & 3 \\ 1 & 2 \end{matrix}, \vec{u_3} = \begin{matrix} -2 & 0 \\ 1 & 3 \end{matrix}$$

$$\vec{v} = c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3}$$

$$\begin{matrix} 0 & 8 \\ 2 & 1 \end{matrix} = \begin{matrix} -c_2 - 2c_3 & 2c_1 + 3c_2 \\ c_1 + c_2 + c_3 & 2c_2 + 3c_3 \end{matrix}$$

$$\begin{matrix} 0 & -1 & -2 & 0 \\ 2 & 3 & 0 & 8 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \end{matrix} \text{, row echelon form: } \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

So $\begin{matrix} 0 & 8 \\ 2 & 1 \end{matrix} = 1 \begin{matrix} 0 & 2 \\ 1 & 0 \end{matrix} + 2 \begin{matrix} -1 & 3 \\ 1 & 2 \end{matrix} - \begin{matrix} -2 & 0 \\ 1 & 3 \end{matrix}$

Spanning sets:

Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a subspace of a vector space V. The set S is called a spanning set of V if every vector in the vector space V can be written as a linear combination of vectors in S. In such cases, we say that S spans V.

Example: The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ spans $\mathbb{R}^3$ since every vector $\vec{u} = (u_1,u_2,u_3) = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$.

$S = \{(1,2,3),(0,1,2),(-1,0,1)\}$
$(u_1 - u_3, 2u_1 + u_2, 3u_1 + u_2 + u_3)$

$$\det\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 0$$

Therefore, S is not a spanning set.

A set of vectors $S = \{v_1, v_2, \ldots, v_R\}$ in a vector space V is called linearly independant if the vector equation $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}$ has only the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If not, then S is linearly dependant.

Example: Determine whether the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ is dependant or not.
$_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}$

$$\begin{matrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{matrix} \text{, row echelon form: } \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

Theorem: A set $S = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\right\}, k \geq 2$ is linearly dependant if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors.
$c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}$
Without loss of generality (WLG), suppose $c_1 \neq 0$.
$c_1\vec{v_1} = -c_2\vec{v_2} - \ldots - c_k\vec{v_k}$, so $\vec{v_1} = -\frac{c_2}{c_1}\vec{v_2} - \ldots - \frac{c_k}{v_k}\vec{v_k}$.
Conversely, if $\vec{v_1} = c_2\vec{v_2} + \ldots + c_k\vec{v_k} \rightarrow \vec{v_1} - c_2\vec{v_2} - \ldots - c_k\vec{v_k} = 0$
Therefore, if a $c_n \neq 0$, then the equation is dependant.

Two vectors are linearly dependant if one is a scalar multiple of the other.
$S = \{(1,1,1),(2,2,2)\}$ is a linearly dependant set.

Section 4.5 - Basis and Dimensions

A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called a b asis for V if the following conditions are true:

1. S spans V
2. S is linearly independent

- A standard basis for $\mathbb{R}^2$ is $\{e_{11}, e_{21}, \ldots, e_n\}$ where $e_i = (0, 0, \ldots, 1, \ldots, 0)$
- A monstandard basis for $\mathbb{R}^2$ is $S = \{(1,2), (2,1)\}$

The standard basis is $\vec{i}, \vec{j}, \vec{k}$

For P_n (polynomials degree $\leq n$), a standard basis is $\{1, x, x^2, \ldots, x^{n-1}, x^n\}$

For $M_{2,2}$, $\left\{ \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}, \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\}$

*Theorem: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a space V. Then every vector in V can be written in one and only one way as linear combinations of vectors in S.

Proof: Let $\vec{u} \in V$. Then, there exist $c_1, c_2, \ldots, c_n : u = c_1v_1 + c_2v_2 + \ldots + c_nv_n$. (Spanning set)

Suppose $u = b_1v_1 + b_2v_2 + \ldots + b_nv_n$. Then $(c_1 - b_1)v_1 + (c_2 - b_2)v_2 + \ldots + (c_n - b_n)v_n = 0$.

But S is a basis, therefore it is linearly independent. So $c_1 - b_1 = c_2 - b_2 = c_n - b_n = 0$. Therefore, $c_i = b_i$ for every $i \in \{1, \ldots, n\}$.

Consequently, the representation is unique.

*Theorem: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for vector space V, then every set containing more than n vectors in V is linearly dependent.

Corollary: If a vector space V has one basis with n vectors, then every basis for the vector space has the same number of elements.

If a vector space V has a basis consisting of n vectors, then the number n is called the dimension of V, denoted by $\dim(V) = n$.

Examples: $\dim(\mathbb{R}^n) = n$; $\dim(M_{n,m}) = m \times n$

$V =$ subspace of symmetric matrices in $M_{2,2}$

$\dim(V) = 3$

basis: $\left\{ \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\}$

Theorem: Let V be a vector space of dimension n.

1. If $S = \{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{v_1, v_2, \ldots, v_n\}$ spans V, then S is a basis for V.

Section 4.6 - Rank of a Matrix and Systems of Linear Equations

Let A be a $m \times n$ matrix.

1. The row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
2. The column space of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

If A is an $m \times n$ matrix, then the row space and column space of A have the same dimensions.

The dimension of the row space or the column space is called the rank of matrix A.

Rank is denoted by $rank(A)$.

Example: Find the rank of the matrix A given by

$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

The dimension is 3, so the rank is 3.

$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$, rank: 3

If A is am $m \times n$ matrix, then the set of all solutions of the homogenous system of linear equations $Ax = 0$ is a subspace of $\mathbb{R}^n$, called the null space of A, denoted by $N(A)$. $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$. The dimension of the null space of A is called the nullity of A.

Example 1: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a null space.

$N(A) = \{(0,0)\}$

nullity(A) $= 0$

Example 2: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(A) = \{(-2t, t), t \in \mathbb{R}\}$

nullity(A) = 1

If A is a $m \times n$ matrix of rank r, then $n = rank(A) + nullity(A)$.

For square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent:

1. A is invertable
2. $\det(A) \neq 0$
3. $Ax = b$ has a unique solution for any $n \times 1$ matrix b which is $x = A^{-1}b$
4. Rank(A) = n
5. nullity(A) = 0
6. The n row vectors of A are linearly independent.
7. The n column vectors of A are linearly independent.