6.1 - Introduction to Linear Transformations

Definition: Let V and W be vector spaces. The function $T : V \to W$ is called a linear transformation of V into W if the following two properties are true for all $\vec{u}$ and $\vec{v}$ in V for any scalar c:

1. $T(u+v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

Example 1: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(v_1 v_2) = (2v_1, v_1 + v_2)$

Let $u = (u_1, u_2),\ v = (v_1, v_2),\ c \in \mathbb{R}$

1. $T(u+v) = T(u_1+v_1, u_2+v_2)$

$= (2(u_1+v_1), (u_1+v_1)+(u_2+v_2))$

$= (2u_1, u_1+u_2) + (2v_1, v_1+v_2)$

$= T(u) + T(v)$

2. $T(cu) = T(c(u_1, u_2))$

$= T(cu_1, cu_2)$

$= (2cu_1, cu_1 + cu_2)$

$= c(2u_1, u_1+u_2)$

$= cT(u_1, u_2) = cT(u)$

Example 2: If $W = V$, then T is a linear operator.

$T : C[0,1] \to C[0,1]$

$f \to f'$

$T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$

$T(cf) = (cf)' = cf' = cT(f)$

Examples of non-linear transformations:

$T : \mathbb{R} \to \mathbb{R}$

$x \to x^2$

$T(x+y) = (x+y)^2 \neq (x^2+y^2) = T(x) + T(y)$

So T is not a linear transformation.

$T : \mathbb{R} \to \mathbb{R}$

$x \to x+1$

$T(x+y) = x+y+1$

$T(x) + T(y) = (x+1) + (y+1) = x+y+2$

T is not a linear transformation.

$T(0) \neq 0$

Theorem: Let T be a linear transformation from V to W, where $\vec{u}$ and $\vec{v}$ are in V, the following properties must be true:
1. $T(0) = 0$
2. $T(-v) = -T(v)$
3. $T(u-v) = T(u) - T(v)$
4. If $v = \sum_{i=1}^{n} c_i v_i$ then $T(v) = \sum_{i=1}^{n} c_i T(v_i)$

Proof:
1. $T(0+0) = T(0) + T(0)$
$T(0) = 2T(0) \Rightarrow 0 = T(0)$
2. $T(-v) = -1T(v) = -T(v)$
3. $T(u-v) = T(u+(-v)) = T(u) + T(-v) = T(u) - T(v)$

$T : \mathbb{R} \to \mathbb{R}$
$x \to x^2$
$T(-x) = x^2 = T(x) \neq -T(x)$
So it is not a linear transformation

Theorem: Let A be an $m \times n$ matrix. The function T defined by $T(v) = Av$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^n$.

Examples:
$\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x+3y \end{bmatrix}$$

$(x,y) \to (x+2y, 2x+3y)$

Let $T : M_{m,n} \to M_{n,m}$
$A \to A^T$
is a linear transformation because $(A+B)^T = A^T + B^T$
$(cA)^T = cA^T$

6.2 - The Kernel and Range of a Linear Transformation

Definition: Let $T : V \to W$ be a linear transformation. Then the set of all vectors $v \in V$ that satisfy $T(v) = 0$ is called the kernel of T and is denoted by $\ker(T)$.

Example 1: $T : R^2 \to R^3$
$(x_1, x_2) \mapsto (x_1 - 2x_2, 0, -x_1)$

$T(x_1, x_2) = (0,0,0)$
Therefore, $(x_1 - 2x_2, 0, -x_1) = (0,0,0)$
$x_1 - 2x_2 = 0$
$x_2 = \frac{x_1}{2} = 0$
$0 = 0$
$-x_1 = 0 \Rightarrow x_1 = 0$
So $(x_1, x_2) = (0,0)$
So $\ker(T) = \{(0,0)\} = \{0\}$

Example 2: Let $T : R^5 \to R^4$
$x \mapsto Ax$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

$T(x) = 0 \Rightarrow Ax = 0$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix}$$

Row echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mapsto$$

$x_1 + 2x_3 - x_5 = 0$
$x_2 - x_3 - 2x_5 = 0$
$x_4 + 4x_5 = 0$
$x_5 = t$
$x_3 = s$
$x_1 = -2x_3 + x_5 = -2s + t$
$x_2 = x_3 + 2x_5 = s + 2t$
$x_4 = -4t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Basis for the kernel: $B = \{(-2,1,1,0,0),(1,2,0,-4,1)\}$

range(T) $= \{T(v : v \in V)\}$
Theorem: The range of a linear transformation $T : V \to W$ is a subspace of W.

Corollary: Let $T : R^n \to R^m$ be a linear transformation given by $T(x) = Ax$. Then the column space of A is equal to the range of T.

The dimension of the kernel of T is called the nullity of T. It is denoted by nullity(T).
The dimension of the range of T is called the rank of T. It is denoted by rank(T).
rank(T) $+$ nullity(T) $=$ n
dim(range) $+$ dim(kernel) $=$ dim(domain)

$R^2 \to R^2$
$(x_1,x_2) \mapsto (2x_1+x_2, x_1-x_2)$
$T(x) = Ax$
$x = (x_1,x_2)^T$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1+x_2 \\ x_1-x_2 \end{bmatrix}$$

A function $T : V \to W$ is called one-to-one (or injective) if and only if for every $u,v \in V$
$T(u) = T(v) \Rightarrow u = v$

Theorem: Let $T : V \to W$ be a linear transformation. Then, T is one-to-one if and only if $\ker(T) = \{0\}$.
*Proof: Suppose T is one-to-one. Then
$T(v) = 0$
$\Rightarrow T(v) = T(0)$
$\Rightarrow v = 0$
$\Rightarrow \ker(T) = \{0\}$
Suppose $\ker(T) = \{0\}$
Let $u,v \in V,\ T(u) = T(v)$

$\Rightarrow T(u) - T(v) = 0$
$\Rightarrow T(u - v) = 0$
As $\ker(T) = \{0\}$ then $u - v = 0 \Rightarrow u = v$
So T is one-to-one

A function $T : V \to W$ is said to be onto (surjective) if W is equal to the range of T.
For every $w \in W$, there exists a $v \in V$ such that $T(v) = w$.

Theorem: Let $T : V \to W$ be a linear transformation, where W is finite dimensional.
Then T is onto if and only if rank(T) = dim(W)

Theorem: Let $T : V \to W$ be a linear transformation with vector spaces
$V, W : \dim(V) = \dim(W) = n$. Then T is one-to-one if and only if it is onto.
*Proof: If T is one-to-one then $\ker(T) = \{0\}$, so the dimension of the kernel is 0.
Therefore, $\dim(range(T)) = n - \dim(\ker(T)) = n - 0 = n$
So $rank(T) = \dim(W)$
Conversely, if T is onto, then $\dim(range(T)) = \dim(W) = n$
So $\dim(\ker(T)) = 0$ and T is one-to-one.

Definition: A linear transformation $T : V \to W$ that is one-to-one and onto is called an isomorphism. Moreover, if V, W are vector spaces such that there exists an isomorphism from V to W, then we say that they are isomorphic to each other.

**Theorem: Two finite-dimensional vector spaces V and W are isomorphic if and only if they are of the same dimension.
Proof: $(\Rightarrow)$ Suppose that V is isomorphic to W, and $\dim(V) = n$.
Therefore, there exists an isomorphism $T : V \to W$ that is one-to-one and onto.
So $\dim(\ker(T)) = 0$ (because T is one-to-one) and $\dim(range(T)) = n = \dim(W)$ (because T is onto).
So $\dim(W) = \dim(V)$
$(\Leftarrow)$ Suppose that $\dim(V) = \dim(W) = n$.
Let $B = \{v_1, v_2, \ldots, v_n\}$
Let $B' = \{w_1, w_2, \ldots, w_n\}$

Let us define $T : V \to W$
$c_1v_1 + c_2v_2 + \ldots + c_nv_n \mapsto c_1w_1 + c_2w_2 + \ldots + c_nw_n$
$T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = T(\alpha_1v_1 + \alpha_2v_2 + \ldots + \alpha_nv_n)$
$c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n) = \alpha_1T(v_1) + \alpha_2T(v_2) + \ldots + \alpha_nT(v_n)$
$c_1w_1 + c_2w_2 + \ldots + c_nw_n = \alpha_1w_1 + \alpha_2w_2 + \ldots + \alpha_nw_n$
So $c_i = \alpha_i$ for $i \in \{1, \ldots, n\}$
and T is one-to-one

and T is onto because $\dim(v) = \dim(W)$

Consequently, T is an isomorphism. So $V \approx W$.

6.3 - Matrices for Linear Transformations

Theorem: Let $T : R^n \to R^m$ be a linear transformation s.t.

$$T(R_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix}, \ T(R_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix}, \ T(R_3) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

Then the $m \times n$ matrix whose columns correspond to $T(c_i)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is such that $T(v) = Av$ for every $v \in R^n$.
A is called the standard matrix for T.

Example: Find the standard matrix for the linear transformation $T : R^2 \to R^2$ defined by
$T(x,y) = \left(x+y,\ x-2y\right)$
$T(1,0) = (1,1)$
$T(0,1) = (1,-2)$
So $A = [T(e_1)|T(e_2)]$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

To check it:

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-2y \end{bmatrix}$$

Example: $D : P_2 \to P_2;\ p \mapsto p'$
$e_1 = 1;\ D(e_1) = 0 = 0e_1 + 0e_2 + 0e_3$
$e_2 = x;\ D(e_2) = 1 = 1e_1 + 0e_2 + 0e_3$
$e_3 = x^2;\ D(e_3) = 2x = 0e_1 + 2e_2 + 0e_3$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Check: $x^2 + 3x + 2$

$D(p) = 2x + 3$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Theorem: Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2. The composition $T : R^n \to R^p$ is defined by $T(v) = T_2(T_1(v))$ is a linear transformation.

Moreover, the standard matrix A for T is given by the matrix product $A = A_2 A_1$

Example: Let T_1, T_2 be linear transformations from $R^3 \to R^3$ such that $T_1(x,y,z) = (2x + y, 0, x + z)$ and $T_2 = (x - y, z, y)$

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

For $T_2 \circ T_1 : \ A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So $T_2 \circ T_1(x,y,z) = (2x + y, x + z, 0)$

Check: $T_2 \circ T_1(x,y,z) = \left(2x + y - 0,\ x + z,\ 0\right)$

Definition: If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are linear transformations such that for every $v \in R^n$, $T_2(T_1(v)) = v$ and $T_1(T_2(v)) = v$, then T_2 is called the inverse of T and we say that T_1 is invertible.

Theorem: Let $T : R^n \to R^n$ be a linear transformation with standard matrix A. Then the following conditions are equivalent:

1. T is invertible
2. T is an isomorphism
3. A is invertible

Moreover, if T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Example: $T : R^3 \to R^3$

$T(x_1,x_2,x_3) = \left(2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3\right)$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$T^{-1}(x_1,x_2,x_3) = \left(-x_1 + x_2,\ -x_1 + x_3,\ 6x_1 - 2x_2 - 3x_3\right)$